

# LINEARLY ORDERED COMPACTS AND CO-NAMIOKA SPACES

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**ABSTRACT.** It is shown that for any Baire space  $X$ , linearly ordered compact  $Y$  and separately continuous mapping  $f : X \times Y \rightarrow \mathbb{R}$  there exists a dense in  $X$   $G_\delta$ -set  $A \subseteq X$  such that  $f$  is jointly continuous at every point of  $A \times Y$ , i.e. any linearly ordered compact is a co-Namioka space.

## 1. INTRODUCTION

Investigation of the joint continuity points set of separately continuous functions defined on the product of a Baire space and a compact space take an important place in the separately continuous mappings theory. The classical Namioka's result [1] become the impulse to the intensification of these investigations. In particular, the following notions were introduced in [2].

Let  $X, Y$  be topological spaces. We say that a separately continuous function  $f : X \times Y \rightarrow \mathbb{R}$  has the *Namioka property* if there exists a dense in  $X$   $G_\delta$ -set  $A \subseteq X$  such that  $f$  is jointly continuous at every point of set  $A \times Y$ .

A compact space  $Y$  is called a *co-Namioka space* if for every Baire space  $X$  each separately continuous mapping  $f : X \times Y \rightarrow \mathbb{R}$  has the Namioka property.

Most general results in the direction of study of co-Namioka space properties were obtained in [3,4]. It was obtained in [3,4] that the class of compact co-Namioka spaces is closed over products and contains all Valdivia compacts. Moreover, it was shown in [3] that the linearly ordered compact  $[0, 1] \times \{0, 1\}$  with the lexicographical order is co-Namioka and it was reproved in [3] (result from [5]) that every completely ordered compact is co-Namioka. Thus, the following question naturally arises: is every linearly ordered compact co-Namioka?

In this paper we give the positive answer to this question.

## 2. DEFINITIONS AND AUXILIARY STATEMENTS

Let  $X$  be a topological space and  $f : X \rightarrow \mathbb{R}$ . For every nonempty set  $A \subseteq X$  by  $\omega_f(A)$  we denote the oscillation

$$\sup\{|f(x') - f(x'')| : x', x'' \in A\}$$

of the function  $f$  on the set  $A$ . Moreover for every point  $x_o \in X$  by  $\omega_f(x_o)$  we denote the oscillation  $\inf\{\omega_f(U) : U \in \mathcal{U}\}$  of the function  $f$  at  $x_o$ , where  $\mathcal{U}$  is the system of all neighborhoods of  $x_o$  in  $X$ .

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Let  $X, Y$  be topological spaces,  $f : X \times Y \rightarrow \mathbb{R}$ ,  $x_o \in X$  and  $y_o \in Y$ . The mappings  $f^{x_o}$  and  $f_{y_o}$  are defined by:  $f^{x_o}(y) = f_{y_o}(x) = f(x, y)$  for every  $x \in X$   $y \in Y$ .

For a linearly ordered set  $(X, <)$  and a point  $x', x'' \in X$ ,  $x' < x''$ , we put

$$[x', x''] = \{x \in X : x' \leq x \leq x''\}, \quad [x', x'') = \{x \in X : x' \leq x < x''\},$$

$$(x', x''] = \{x \in X : x' < x \leq x''\} \text{ and } (x', x'') = \{x \in X : x' < x < x''\}.$$

Points  $x', x'' \in X$ ,  $x' < x''$  is called *neighbor*, if  $(x', x'') = \emptyset$ . Recall that all nonempty open intervals  $(x', x'')$  and intervals  $[a, x)$  and  $(x, b]$ , where  $a$  and  $b$  are the minimal and maximal elements in  $X$  respectively (if they exist) form a base of the topology on  $X$ . It easy to see that a linearly ordered space  $X$  is a compact space with respect to the topology generated by the linear order, if and only if every nonempty closed set  $A \subseteq X$  has in  $X$  minimal and maximal elements.

A topological space  $X$  is called *connected*, if  $A \cup B \neq X$  for every disjoint nonempty open in  $X$  sets  $A$  and  $B$ . Note that a linearly ordered compact  $X$  is connected if and only if  $X$  has not neighbor elements.

**Proposition 2.1.** *Let  $(X, <)$  be a linearly ordered compact,  $f : X \rightarrow \mathbb{R}$  be a continuous function and  $\varepsilon > 0$ . Then there exists  $n \in \mathbb{N}$  such that for every elements  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in X$  such that  $a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n$ , there exists  $k \leq n$  such that  $|f(a_k) - f(b_k)| < \varepsilon$ .*

*Proof.* Consider a finite cover  $(U_i : i \in I)$  of compact space  $X$  by intervals  $U_i$  such that  $\omega_f(U_i) < \varepsilon$  for every  $i \in I$ . Put  $n = |I| + 1$ . Then for every points  $a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n$  from the space  $X$  there exists  $i_o \in I$  such that the interval  $U_{i_o}$  coincides at least three of these points. Taking into account that  $U_{i_o}$  is an interval we obtain that there exists  $k \leq n$  such that  $a_k, b_k \in U_{i_o}$ . Then

$$|f(a_k) - f(b_k)| \leq \omega_f(U_{i_o}) < \varepsilon.$$

□

**Proposition 2.2.** *Let  $(X, <)$  be a linearly ordered connected compact,  $a = \min X$ ,  $b = \max X$ ,  $f : X \rightarrow \mathbb{R}$  be a continuous mappings and  $f(a) \neq f(b)$ . Then there exists a point  $c \in (a, b)$  such that  $f(c) = \frac{1}{2}(f(a) + f(b))$ .*

*Proof.* We put  $y_o = \frac{1}{2}(f(a) + f(b))$ ,  $A = f^{-1}((-\infty, y_o))$  and  $B = f^{-1}((y_o, +\infty))$ . Since  $f$  is continuous, the sets  $A$  and  $B$  are open in  $X$ . It follows from the connectivity of  $X$  that the set  $C = X \setminus (A \cup B)$  is nonempty. It remains to chose a point  $c \in C$ . □

### 3. MAIN RESULTS

**Theorem 3.1.** *Every linearly connected ordered compact is co-Namioka space.*

*Proof.* Let  $X$  be a Baire space,  $(Y, <)$  be a linearly ordered connected compact and  $f : X \times Y \rightarrow \mathbb{R}$  be a separately continuous function. We prove that  $f$  has Namiola property.

Fix  $\varepsilon > 0$  and show that the open set

$$G_\varepsilon = \{x \in X : \omega_f(x, y) < 9\varepsilon \text{ for every } y \in Y\}$$

is dense in  $X$ .

Let  $U$  be a nonempty open in  $X$  set. For every  $x \in U$  we denote by  $N_x$  the set of all  $n \in \mathbb{N}$  such that there exist  $a_1, \dots, a_n, b_1, \dots, b_n \in Y$  such that

$$a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n \text{ and } |f(x, a_i) - f(x, b_i)| > \varepsilon$$

for every  $i = 1, \dots, n$ . It follows from Proposition 2.1 that all sets  $N_x$  are upper bounded. For every  $x \in U$  we put  $\varphi(x) = \max N_x$ , if  $N_x$  is nonempty, and  $\varphi(x) = 0$ , if  $N_x = \emptyset$ . It follows from the continuity of  $f$  with respect to the first variable that for every  $n$  the set  $\{x \in U : \varphi(x) > n\}$  is open in  $U$ , i.e. the function  $\varphi : U \rightarrow \mathbb{Z}$  is upper semicontinuous on the Baire space  $U$ . Therefore (see [6]) the function  $\varphi$  is pointwise discontinuous. Then there exist an open in  $U$  nonempty set  $U_o$  and nonnegative  $n \in \mathbb{Z}$  such that  $\varphi(x) = n$  for every  $x \in U_o$ .

If  $n = 0$ , then  $|f(x, a) - f(x, b)| \leq \varepsilon$  for every  $x \in U_o$ ,  $a, b \in Y$ . Then taking any point  $y_o \in Y$  and an open nonempty set  $U_1 \subseteq U_o$  such that  $\omega_{f_{y_o}}(U_1) < \varepsilon$ , we obtain that  $\omega_f(x, y) < 3\varepsilon$  for every  $x \in U_1$ ,  $y \in Y$ . In particular,  $U_1 \subseteq G_\varepsilon$ .

Now we consider the case of  $n \in \mathbb{N}$ . We take a point  $x_o \in U_o$  and choose  $a_1, \dots, a_n, b_1, \dots, b_n \in Y$  such that  $a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n$  and  $|f(x_o, a_i) - f(x_o, b_i)| > \varepsilon$  for  $1 \leq i \leq n$ .

Using the continuity of  $f$  with respect to the first variable we choose an open neighborhood  $U_1 \subseteq U_o$  of  $x_o$  in  $U$  such that  $|f(x, a_i) - f(x, b_i)| > \varepsilon$  for every  $x \in U_1$  and  $i \in \{1, \dots, n\}$ . Show that for every  $y_o \in Y$  there exists an open neighborhood  $V$  of  $y_o$  in  $Y$  such that  $\omega_{f^x}(V) \leq 4\varepsilon$  for every  $x \in U_1$ .

Let  $y_o \in G = Y \setminus \bigcup_{i=1}^n [a_i, b_i]$ . Since the set  $G$  is open in  $Y$ , there exists an open in  $Y$  interval  $V$  such that  $V \subseteq G$ . Then for every  $a, b \in V$  with  $a < b$  we have  $[a, b] \cap [a_i, b_i] = \emptyset$  for every  $i = 1, \dots, n$ . Taking into account that  $\varphi(x) = n$  and  $|f(x, a_i) - f(x, b_i)| > \varepsilon$  for every  $x \in U_1$  and  $i \in \{1, \dots, n\}$  we obtain that  $|f(x, a) - f(x, b)| \leq \varepsilon$ , i.e.  $\omega_{f^x}(V) \leq \varepsilon$  for every  $x \in U_1$ .

Let  $a_i < y_o < b_i$  for some  $i \in \{1, \dots, n\}$ . Then we put  $V = (a_i, b_i)$ . Note that  $|f(x, a) - f(x, b)| \leq 2\varepsilon$  for every points  $a, b \in (a_i, b_i)$  and  $x \in U_1$ . Really, we suppose that there exist  $a, b \in (a_i, b_i)$  and  $x \in U_1$  such that  $|f(x, a) - f(x, b)| > 2\varepsilon$ . Then according to Proposition 2.2 there exist a point  $c \in (a, b)$  such that  $|f(x, a) - f(x, c)| = |f(x, c) - f(x, b)| > \varepsilon$ . But this contradicts to  $\varphi(x) = n$ . Thus,  $\omega_{f^x}(V) \leq 2\varepsilon$ .

It remains to consider the case of  $y_o \in \{a_i, b_i : 1 \leq i \leq n\}$ . Let  $a_o = \min Y$ ,  $b_o = \max Y$  and  $a_o < y_o < b_o$ . Put  $y_1 = \max(\{a_i, b_i : 0 \leq i \leq n\} \cap [a_o, y_o])$ ,  $y_2 = \min(\{a_i, b_i : 0 \leq i \leq n\} \cap [y_o, b_o])$  and  $V = (y_1, y_2)$ . It follows from Proposition 2.2 that for every  $a \in (y_1, y_o]$ ,  $b \in [y_o, y_2)$  and  $x \in U_1$  the following inequalities  $|f(x, a) - f(x, y_o)| \leq 2\varepsilon$  and  $|f(x, y_o) - f(x, b)| \leq 2\varepsilon$  hold. Therefore  $\omega_{f^x}(V) \leq 4\varepsilon$  for every  $x \in U_1$ . In the case  $y_o = a_o$  or  $y_o = b_o$ , it enough to put  $V = [y_o, y_2)$  or  $V = (y_1, y_o]$ .

Now we prove that  $U_1 \subseteq G_\varepsilon$ . Let  $(x_o, y_o) \in U_1 \times Y$ . Take an open neighborhood  $V$  of  $y_o$  in  $Y$  such that  $\omega_{f^x}(V) \leq 4\varepsilon$  for every  $x \in U_1$ . Using the continuity of  $f$  with respect to the first variable we choose a neighborhood  $U_2 \subseteq U_1$  of  $x_o$  in  $X$  such that  $\omega_{f_{y_o}}(U_2) < \varepsilon$ . Then  $\omega_f(U_2 \times V) < 9\varepsilon$ .

Thus for every  $\varepsilon$  we have that the set  $G_\varepsilon$  is dense in Baire space  $X$ . Then the  $G_\delta$ -set  $A = \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$  is dense in  $X$  too. Moreover, the function  $f$  is jointly continuous at every point of set  $A \times Y$ , i.e.  $f$  has Namioka property.  $\square$

**Corollary 3.2.** *Every linearly ordered compact is a co-Namioka space.*

*Proof.* Let  $(Y, <)$  be a linearly ordered compact. If the set  $D$  of all pairs of neighbor points in  $Y$  is empty, then according to Theorem 3.1 the space  $Y$  is co-Namioka.

Let  $D \neq \emptyset$ . For every pair  $d \in D$  of neighbor points in  $Y$  we denote by  $a_d$  and  $b_d$  the left and the right neighbor points of the pair  $d$ , i.e.  $d = \{a_d, b_d\}$  and  $a_d < b_d$ . We put  $Z = Y \cup (D \times (0, 1))$  and define a linear order on  $Z$ , which is an extension of the order on  $Y$ . Let  $z' \in Y$ ,  $d \in D$ ,  $t \in (0, 1)$  and  $z'' = (d, t)$ . Then  $z' < z''$ , if  $z' \leq a_d$ , and  $z'' < z'$ , if  $b_d \leq z'$ . Let  $z' = (d', t')$ ,  $z'' = (d'', t'') \in D \times (0, 1)$ . Then  $z' < z''$ , if  $a_{d'} < a_{d''}$ , or  $d' = d''$  and  $t' < t''$ .

Note that  $(Z, <)$  is a compact space which has not neighbor elements. Moreover the space  $(Y, <)$  is a compact subspace of  $(Z, <)$ .

Let  $X$  be a Baire space and  $f : X \times Y \rightarrow \mathbb{R}$  be a separately continuous function. We construct a mapping  $g : X \times Z \rightarrow \mathbb{R}$ , which is an extension of  $f$ . For every  $x \in X$  and  $z = (d, t) \in D \times (0, 1)$  we put  $g(x, z) = (1-t)f(x, a_d) + tf(x, b_d)$ . It is easy to see that the function  $g$  is separately continuous too. Therefore according to Theorem 3.1 there exists a dense in  $X$   $G_\delta$ -set  $A \subseteq X$  such that  $g$  is jointly continuous at every point of set  $A \times Z$ . Therefore the function  $f$  is jointly continuous at every point of the set  $A \times Y$ . Thus,  $f$  has Namioka property and  $Y$  is co-Namioka.  $\square$

#### 4. EXAMPLE

In this section we show that condition of continuity of  $f$  with respect to the first variable in the above reasoning can not be replaced by quasicontinuity.

Let  $X$  be a topological space and  $f : X \rightarrow Y$ . Recall that the mapping  $f$  is called *quasicontinuous at a point*  $x_o \in X$ , if for every neighborhoods  $U$  of  $x_o$  in  $X$  and  $V$  of  $y_o = f(x_o)$  in  $Y$  there exists an open in  $X$  nonempty set  $G \subseteq U$  such that  $f(G) \subseteq V$ . A mapping  $f$  is called *quasicontinuous*, if  $f$  is quasicontinuous at every point  $x \in X$ .

**Example 4.1.** Let  $X = (0, 1)$  and  $Y = [0, 1] \times \{0, 1\}$  be a linearly ordered compact with the lexicographical order, i.e.  $(y, i) < (z, j)$ , if  $y < z$  or  $y = z$  and  $i < j$ . For every  $t \in [0, 1]$  we put  $t_l = (t, 0)$  and  $t_r = (t, 1)$ . The function  $f : X \times Y \rightarrow \mathbb{R}$  we define by:  $f(x, y) = 0$ , if  $x_r \leq y$ , and  $f(x, y) = 1$ , if  $x_l \geq y$ .

For every  $x \in (0, 1)$  we have  $(f^x)^{-1}(0) = [x_r, 1_r]$  and  $(f^x)^{-1}(1) = [0_l, x_l]$ . Therefore all functions  $f^x$  are continuous. If  $y \in \{0_l, 0_r\}$ , then  $f_y(x) = 1$  for every  $x \in X$ , and if  $y \in \{1_l, 1_r\}$ , then  $f_y(x) = 0$  for every  $x \in X$ . Let  $z \in (0, 1)$ . Then for  $y = z_l$  we have  $f_y(x) = 0$ , if  $x \in (0, z)$ , and  $f_y(x) = 1$ , if  $x \in [z, 1)$ . And for  $y = z_r$  we have  $f_y(x) = 0$ , if  $x \in (0, z]$ , and  $f_y(x) = 1$ , if  $x \in (z, 1)$ . Thus, the function  $f$  is quasicontinuous with respect to the first variable. But the function  $f$  is jointly discontinuous at every point  $(x, x_l)$  and  $(x, x_r)$  for  $x \in X$ .

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